Modified Gravity

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Summary

- What is modified gravity (MG)?
- Relevance: regimes, cosmology.
- Conditions to be imposed on a MG.
- Theories with extra fields: scalar-tensor and generalizations.
- Variational principles: the example of GR.
- Higher derivative theories: metric $f(R)$ and generalizations.
- $f(\mathscr{R})$ theories in the Palatini formalism.

- **What is modified gravity?**

GR: spacetime is described by a manifold endowed with a metric tensor and a symmetric connection derived from it. The metric tensor obeys the EOM following from

$$I = (16\pi G)^{-1} \int R(-g)^{1/2}\, d^4x + I_{\rm m}(\psi_{\rm m}, g_{\mu\nu}),$$

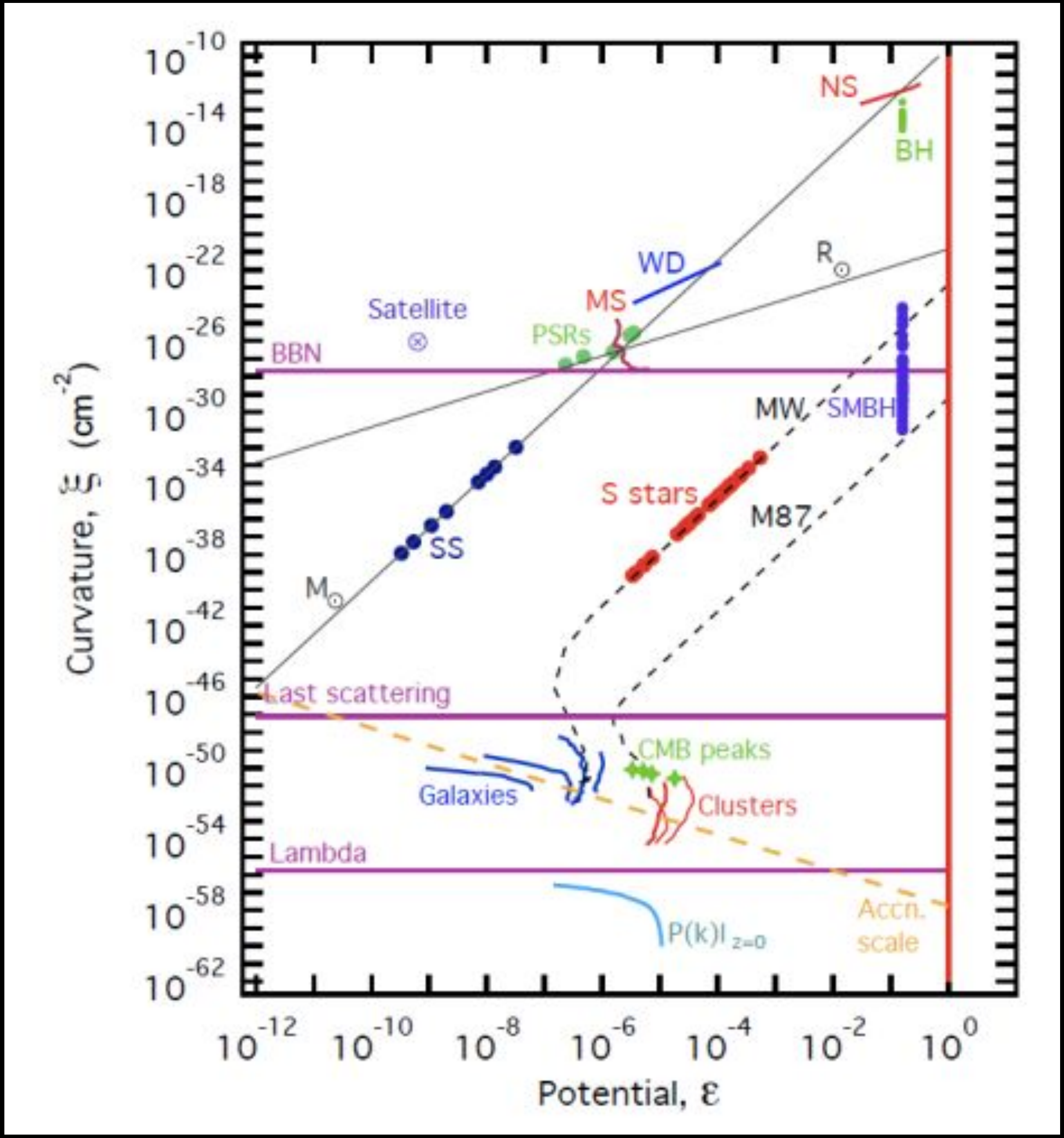
1) **The metric is the sole field associated to gravity.**
2) **Matter fields are universally coupled to the metric.**
3) **The EOM**

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \qquad T^{\mu\nu}{}_{;\nu} = 0$$

are PDEs of 2nd order in the metric, and do not display derivatives of the energy-momentum tensor.

Operational definition: a theory that does not satisfy some (or all of) these conditions will be called here a MG theory.

- **In what regimes may MG be relevant?**



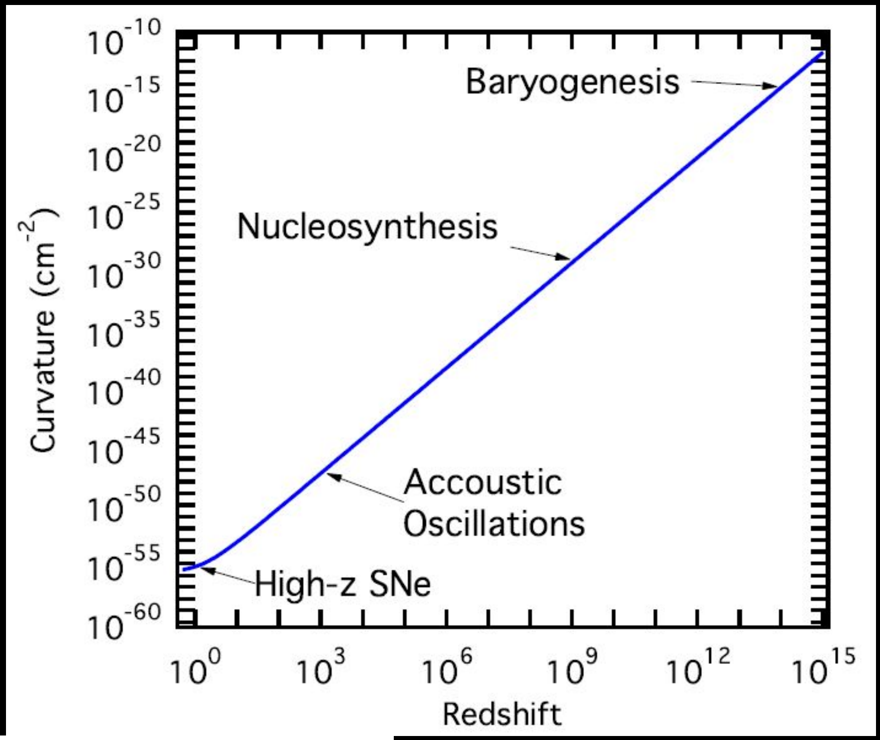
(Baker 2015)

Strong-curvature regime (neutron stars, black holes, early universe).

Weak-curvature regime (accelerated expansion of the universe).

Observations in both regimes are scant but this will change in the near future.

MG may be relevant in the early universe...



Psaltis (2008)

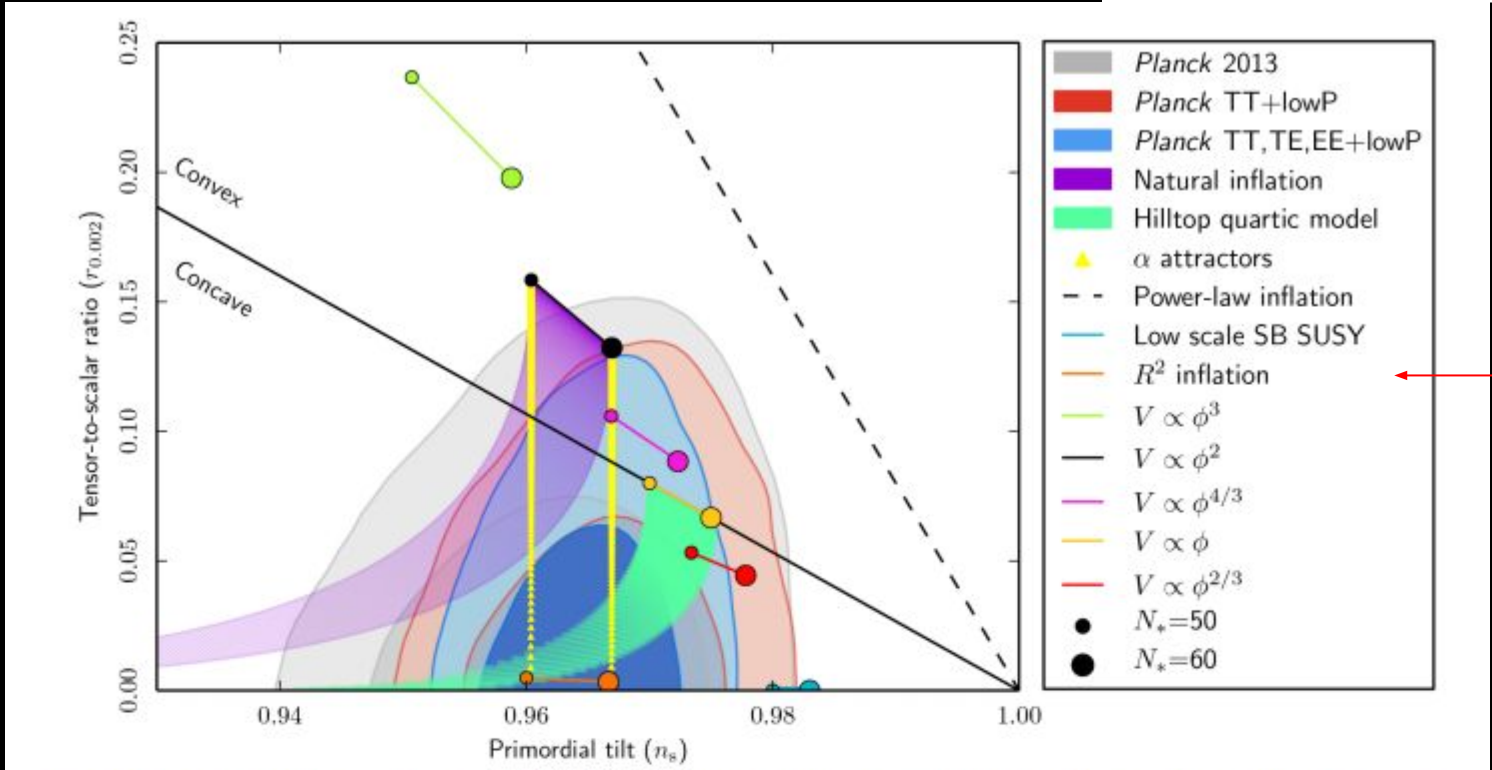


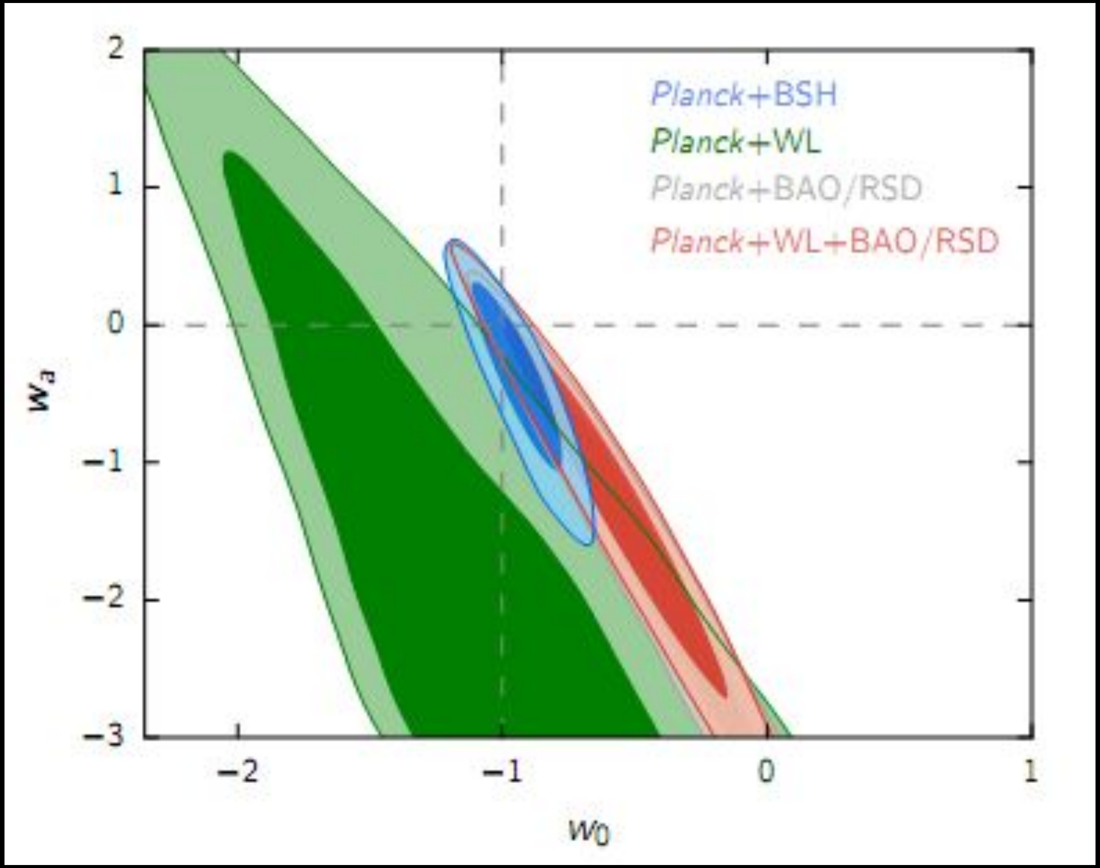
Fig. 12. Marginalized joint 68% and 95% CL regions for n_s and r at $k = 0.002\,\mathrm{Mpc}^{-1}$ from *Planck* compared to the theoretical predictions of selected inflationary models. Note that the marginalized joint 68% and 95% CL regions have been obtained by assuming $\mathrm{d}n_s/\mathrm{d}\ln k = 0$.

Starobinsky (1980).

Planck Collab. (2015).

But we will be exclusively concerned here with the low-curvature regime, and with the "accelerated expansion of the universe" in particular.

$$w(a) = w_0 + (1 - a)w_a \,.$$



Planck Collab. (2015).

Accelerated expansion in GR

$$G_{\mu\nu} = \frac{8\pi G}{3} T_{\mu\nu} \qquad ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right)$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

Assuming that ρ > *0* and *p* = $\omega\,\rho$, in order to get accelerated expansion we need ω < - *1/3* .

How to produce the accelerated expansion?

$$I = (16\pi G)^{-1} \int R(-g)^{1/2}\, d^4x + I_{\rm m}(\psi_{\rm m}, g_{\mu\nu}),$$

1) Modifications of the matter part of the action ($\omega < -1/3$):

- **The cosmological constant:**

$$T^{\Lambda}_{\mu\nu} = -\Lambda g_{\mu\nu}$$

It fits the data but

a) **There is a huge mismatch between the value obtained from cosmology and that calculated from QFT:**

$$\rho_{vac}^{(QFT)} \approx 10^{112}\,\mathrm{erg/cm^3} \qquad \rho_{vac}^{(OBS)} \approx 10^{-8}\,\mathrm{erg/cm^3}$$

b) **If inflation took place, it ended a long time ago, so it must have been fuelled by an effective cosmological constant.**

How to produce the accelerated expansion?

- **Quintessence**

$$I = (16\pi G)^{-1} \int R(-g)^{1/2} d^4x + I_m(\psi_m, g_{\mu\nu}) + I_\phi(\phi, g_{\mu\nu})$$

$$\mathcal{L}_\phi = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

Many models, difficult to get the right potential from particle physics.

- **K-essence, perfect fluids with nontrivial EOS...**

2) Modifications of the gravitational part of the action.

$$I = (16\pi G)^{-1} \int R(-g)^{1/2}\, d^4x + I_{\rm m}(\psi_{\rm m}, g_{\mu\nu}),$$

Some requirements that a MG theory should satisfy:

- **Stability of the relevant solutions.**
- **Agreement with observations in the Solar System (bending of light, anomalous perihelion, Nordtvedt effect, Lens-Thirring effect).**
- **Suppression of short-range modifications of the $1/r^2$ law ("Yukawa potentials").**
- **Agreement with the direct observations of GWs and the variation of the period of binary pulsars.**
- **Agreement with cosmological observations.**
- **…**

By now there are several thousand papers dealing with different MG theories, their consistence and possible observational consequences. I shall present here an overview of some theories that can describe a phase of accelerated expansion, along with a few features of each of them.
Reviews with details about particular theories will be cited along the talk.
f(R) theories in the metric formalism will be given more attention.

Review: Modified gravity and cosmology, T. Clifton et al, Physics Reports, 513, 1 (2012) (aprox. 1300 refs).

- **MG theories with extra fields**

Scalar, vector or tensor fields could be added to the gravitational sector, in such a way that

1) **The metric is the sole field associated to gravity.** ×

2) **Matter fields are universally coupled to the metric.** ✔

3) **The EOM are PDEs of 2nd order in the metric, and do not display derivatives of the metric tensor.** ✔

a) Scalar-tensor theories ("extended quintessence")

$$\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla^\mu\phi - 2\Lambda(\phi)\right] + \mathcal{L}_m(\Psi, g_{\mu\nu}),$$

- **Generalization of Brans-Dicke theory, in which ω = constant.**

- **$\omega(\phi)$ and $\Lambda(\phi)$ are arbitrary functions.**

$$\phi G_{\mu\nu} + \left[\Box\phi + \frac{1}{2}\frac{\omega}{\phi}(\nabla\phi)^2 + \Lambda\right] g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi - \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi = 8\pi T_{\mu\nu}.$$

$$(2\omega + 3)\Box\phi + \omega'(\nabla\phi)^2 + 4\Lambda - 2\phi\Lambda' = 8\pi T.$$

- **Ordinary matter with non-zero trace is the source of ϕ:**

$$(2\omega + 3)\Box\phi + \omega'(\nabla\phi)^2 + 4\Lambda - 2\phi\Lambda' = 8\pi T.$$

- **The possibility of generating an accelerated expansion follows from rewriting**

$$\phi G_{\mu\nu} + \left[\Box\phi + \frac{1}{2}\frac{\omega}{\phi}(\nabla\phi)^2 + \Lambda\right] g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi - \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi = 8\pi T_{\mu\nu}.$$

as

$$G_{\mu\nu} = \frac{8\pi}{\phi}T_{\mu\nu} + T^{(\phi)}_{\mu\nu}$$

and reconstructing the functions $\omega(\phi)$ and $\Lambda(\phi)$ in such a way that, for the case of the EOM with the FLRW metric, $p^{(\phi)} \approx -\rho^{(\phi)}$.

Capozziello et al (2007).

A family of these models satisfying both local gravity and cosmological constraints was studied in Tsujikawa et al (2008).

- **Generalized scalar-tensor theories**

The most general action of a scalar field coupled to gravity that leads to 2nd order EOM is given by

$$S=\sum_{i=2}^{5} S_i=\sum_{i=2}^{5}\int \mathrm{d}^4x\sqrt{-g}\,\mathcal{L}_i\,,$$

$$\begin{aligned}
\mathcal{L}_2 &= K(\varphi, X),\\
\mathcal{L}_3 &= -G_3(\varphi, X)\Box\varphi,\\
\mathcal{L}_4 &= G_4(\varphi, X)R + G_{4,X}\left[(\Box\varphi)^2 - (\nabla_\mu\nabla_\nu\varphi)(\nabla^\mu\nabla^\nu\varphi)\right],\\
\mathcal{L}_5 &= G_5(\varphi, X)\,G_{\mu\nu}(\nabla^\mu\nabla^\nu\varphi)\\
&\quad -\frac{1}{6}G_{5,X}\left[(\Box\varphi)^3 - 3(\Box\varphi)(\nabla_\mu\nabla_\nu\varphi)(\nabla^\mu\nabla^\nu\varphi) + 2(\nabla^\mu\nabla_\alpha\varphi)(\nabla^\alpha\nabla_\beta\varphi)(\nabla^\beta\nabla_\mu\varphi)\right].
\end{aligned}$$

$X = -1/2\, g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi$.

***K, G_3, G_4,* and *G_5* are arbitrary functions.**

This result was actually derived by Horndeski in 1974 , and rederived by Deffayet et al (2011).

The term with non-minimal coupling to the Einstein tensor is needed to remove some of the 3rd and 4th order derivatives of the scalar field (see Deffayet 2013).

Although it is hard to work with these models due to the complexity of the EOM and the arbitrariness in the functions K and G_i, there have been many partially successful attempts to set limits on the arbitrary functions, in order to describe the accelerated expansion and some of the conditions mentioned before. See Kennedy et al (2017), and Alonso et al (2017).

- **Variational principles: the example of GR.**

i) Metric formulation: spacetime described by a manifold with a metric and the Christoffel symbols as connection (which are symmetric, hence torsion-less):

$$\Gamma^{\mu}{}_{\alpha\nu} = \frac{1}{2} g^{\mu\lambda} (g_{\lambda\alpha,\nu} + g_{\lambda\nu,\alpha} - g_{\alpha\nu,\lambda})$$

Einstein´s eqns. follow from the variation of

$$S = \frac{1}{16\pi G} \int \sqrt{-g} (R - 2\Lambda) d^4x + \int \mathcal{L}_m (g_{\mu\nu}, \psi) d^4x,$$

wrt to the metric:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - g_{\mu\nu}\Lambda.$$

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g_{\mu\nu}}.$$

- **Variational principles: the example of GR.**

ii) Palatini formulation

$$S=\frac{1}{16\pi G}\int\sqrt{-g}\,[g^{\mu\nu}\ {}^{\Gamma}R_{\mu\nu}-2\Lambda]\,d^4x+\int\mathcal{L}_m(g_{\mu\nu},\psi)d^4x,$$

where matter does not couple to the connection, which is at this stage assumed to be independent of the metric, and

$${}^{\Gamma}R_{\mu\nu}=\partial_\alpha\Gamma^{\alpha}{}_{\mu\nu}-\partial_\mu\Gamma^{\alpha}{}_{\alpha\nu}+\Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\alpha}{}_{\mu\nu}-\Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\alpha\nu}.$$

Taking the variation of *S* wrt to the connection (assumed to be torsion-less, hence symmetric), it follows that

(Hehl & Kerlick I, 1978)

$$\Gamma^{\mu}{}_{\alpha\nu}=\frac{1}{2}g^{\mu\lambda}(g_{\lambda\alpha,\nu}+g_{\lambda\nu,\alpha}-g_{\alpha\nu,\lambda})$$

Einstein´s eqns. follow from the variation of *S* wrt to the metric:

$$R_{\alpha\beta}-\frac{1}{2}Rg_{\alpha\beta}+\Lambda g_{\alpha\beta}=0.$$

- **Starting with the Einstein-Hilbert Lagrangian, both variational procedures lead to the EOM of GR. This is not the case for other lagrangians, for example those corresponding to *f(R)* theories (Buchdal, 1970).**

- **LIfting the conditions that the matter fields do not couple to the connection and that the torsion is zero leads to the so-called metric-affine theories of gravity, which will not be discussed here, see Hehl & Kerlick II, 1978.**

- **Higher-derivative gravity**

1) **The metric is the sole field associated to gravity. ×**

2) **Matter fields are universally coupled to the metric. ✔**

3) **The EOM are PDEs of 2nd order in the metric ×, and do not display derivatives of the metric tensor ✔.**

1) *f(R)* in the metric formalism

$$S_{met} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R)$$

Taking the variation wrt the metric,

4th order terms

$$\frac{df(R)}{dR}R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]\frac{df(R)}{dR} = \kappa T_{\mu\nu}$$

Trace:

$$\frac{df(R)}{dR}R - 2f(R) + 3\Box\frac{df(R)}{dR} = \kappa T$$

Extra (scalar) degree of freedom.

The covariant derivative is definided with the metric connection.

RG:

$$R = -\kappa T$$

Equivalence between the metric *f(R)* formalism and a scalar-tensor theory

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, [f(\chi) + f_{,\chi}(\chi)(R - \chi)]$$

Taking the variation wrt χ: $f_{,\chi\chi}(\chi)(R - \chi) = 0$

If $f_{,\chi\chi}(\chi) \neq 0$ → $\chi = R$

Defining $\varphi \equiv f_{,\chi}(\chi)$ → $\chi = f_{,\chi}^{-1}(\varphi)$

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa}\varphi R - U(\varphi)\right]$$

$$U(\varphi) = \frac{\chi(\varphi)\,\varphi - f(\chi(\varphi))}{2\kappa}$$

B-D:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}\varphi R - \frac{\omega_{\mathrm{BD}}}{2\varphi}(\nabla\varphi)^2 - U(\varphi)\right]$$

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2\kappa}\varphi R - U(\varphi)\right]$$

$$\omega_{\mathrm{BD}} = 0$$

("Jordan frame")

EOM:

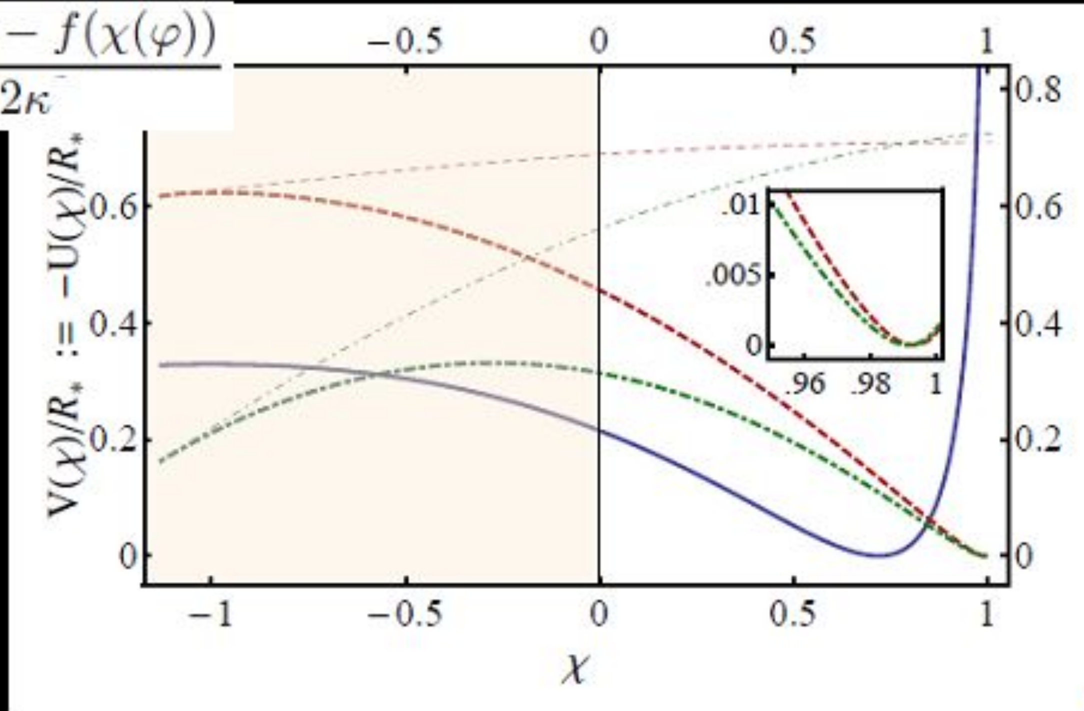
$$G_{\mu\nu} = \frac{\kappa}{\varphi}T_{\mu\nu} - \frac{1}{2\varphi}g_{\mu\nu}U(\varphi) + \frac{1}{\varphi}(\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\Box\varphi)$$

$$3\Box\varphi + 2U(\varphi) - \varphi\frac{dU}{d\varphi} = \kappa T$$

DOF: metric tensor and a scalar.

(Miranda et al, 2009)

$$U(\varphi) = \frac{\chi(\varphi)\,\varphi - f(\chi(\varphi))}{2\kappa}$$



$$f(R) = R - \alpha R_* \ln\left(1 + \frac{R}{R_*}\right)$$

V.Miranda, S. Jorás, I. Waga, and M. Quartin, Phys. Rev. Lett. 102, 221101 (2009)

$$f(R) = R + \lambda R_0 \left(\left(1 + \frac{R^2}{R_0^2}\right)^{-n} - 1\right)$$

A. Starobinsky, JETP Lett. 86, 157 (2007)

$$f(R) = -m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1}$$

W. Hu, I. Sawicki, Phys.Rev .D 76, 064004 (2007)

Metric formalism- the "g-R representation"

L. G. Jaime, L. Patino, M. Salgado
Phys.Rev.D83:024039,2011

Due to the possible problems with the transformation to the BD representation, It would be convenient to have schemes that integrate the original EOM.

$$f_R R_{ab} - \frac{1}{2} f g_{ab} - (\nabla_a \nabla_b - g_{ab} \Box) f_R = \kappa T_{ab} \,,$$

$$f_R := \partial_R f$$

→

$$f_R G_{ab} - f_{RR} \nabla_a \nabla_b R - f_{RRR} (\nabla_a R)(\nabla_b R) + g_{ab} \left[\frac{1}{2} (R f_R - f) + f_{RR} \Box R + f_{RRR} (\nabla R)^2 \right] = \kappa T_{ab}$$

↓

$$G_{ab} = \frac{1}{f_R} \Big[f_{RR} \nabla_a \nabla_b R + f_{RRR} (\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6} \Big(R f_R + f + 2\kappa T \Big) + \kappa T_{ab} \Big] \,.$$

$$\Box R = \frac{1}{3 f_{RR}} \left[\kappa T - 3 f_{RRR} (\nabla R)^2 + 2f - R f_R \right]$$

(The definition of R in terms of $g_{\mu\nu}$ and its derivatives is satisfied identically).

The "Einstein frame"

Metric f(R):

$$S_{\text{met}} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}[\phi R - V(\phi)] + S_M(g_{\mu\nu},\psi).$$

Using the transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f'(R) g_{\mu\nu} \equiv \phi g_{\mu\nu}$

$\phi = f'(R) \to \tilde{\phi}$

$$d\tilde{\phi} = \sqrt{\frac{2\omega_0+3}{2\kappa}}\frac{d\phi}{\phi},$$

$\omega_0 = 0$

$$\phi \equiv f'(R) = e^{\sqrt{2\kappa/3}\tilde{\phi}},$$

$$S'_{\text{met}} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2\kappa} - \frac{1}{2}\partial^\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} - U(\tilde{\phi})\right] + S_M(e^{-\sqrt{2\kappa/3}\tilde{\phi}}\tilde{g}_{\mu\nu},\psi).$$

$$U(\tilde{\phi}) = \frac{Rf'(R) - f(R)}{2\kappa(f'(R))^2},$$

$R = R(\tilde{\phi}),$

which is GR plus a scalar field with a given potential, and non-minimally coupled coupled to matter.

Note however that the initial theory is not conformally invariant, so the new representation should be taken used as a means to (sometimes) simplify the mathematics of the problem.

How to obtain the accelerated expansion from an *f(R)* theory with ordinary matter?

"Curvature quintessence", Capozziello et al (2003).

$$f'(R)R_{\mu\nu} - \tfrac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]f'(R) = \kappa T_{\mu\nu},$$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu},$$

$$H^2 = \frac{\kappa}{3f'}\left[\rho + \frac{Rf' - f}{2} - 3H\dot{R}f'\right],$$

$$2\dot{H} + 3H^2 = -\frac{\kappa}{f'}\left[P + (\dot{R})^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')\right].$$

Defining:

$$\rho_{\rm eff} = \frac{Rf' - f}{2f'} - \frac{3H\dot{R}f''}{f'},$$

$$P_{\rm eff} = \frac{\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')}{f'},$$

If the geometrical contribution dominates over the rest:

$$H^2 = \frac{\kappa}{3}\rho_{\rm eff}, \qquad \frac{\ddot{a}}{a} = -\frac{\kappa}{6}[\rho_{\rm eff} + 3P_{\rm eff}].$$

(with $\rho_{eff} > 0$)

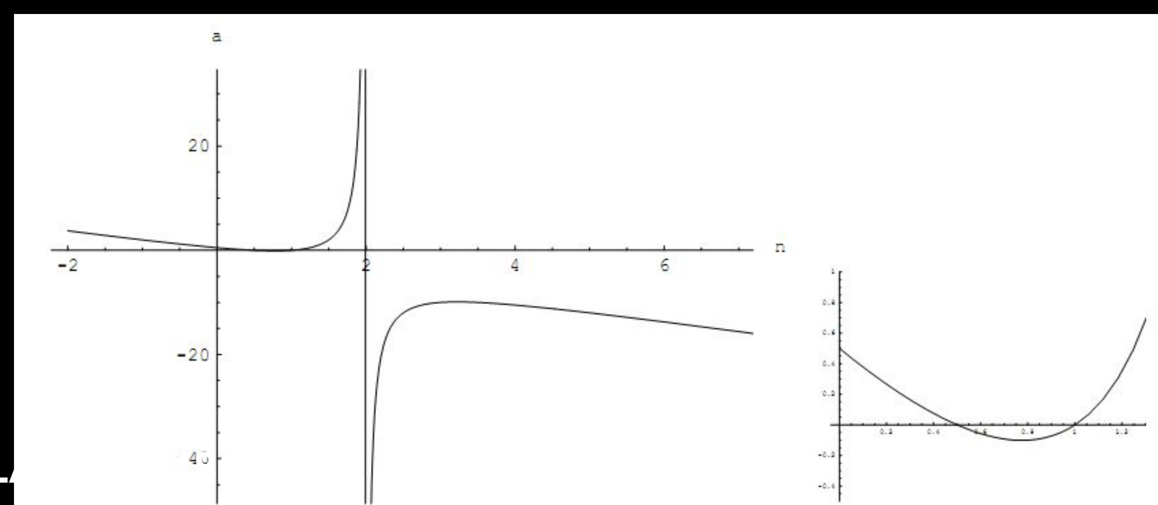
$$w_{\rm eff} \equiv \frac{P_{\rm eff}}{\rho_{\rm eff}} = \frac{\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')}{(Rf' - f)/2 - 3H\dot{R}f''}.$$

The sign of w_{eff} is determined by P_{eff} .

$$f(R) = f_0 R^n\,, \qquad a(t) = a_0\left(\frac{t}{t_0}\right)^{\alpha}$$

$\alpha \geq 1$ **In order to have acc. expansion.**

$$\alpha = \frac{2n^2 - 3n + 1}{2 - n}, \ \forall n \ \text{ but } \ n \neq 2\,.$$



Capozziello et al (2003)

LA

Conditions on the *f(R)* - metric formalism

1)

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{\kappa T_{\mu\nu}}{f'(R)} + g_{\mu\nu} \frac{[f(R) - R f'(R)]}{2 f'(R)} + \frac{[\nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R)]}{f'(R)}$$

The condition $f' > 0$ guarantees that the coupling between gravity and matter is positive.

2) $f(R) \rightarrow R + \Lambda$ for $R >> R_0$ guarantees that GR is recovered in the Early universe.

3) The theory should describe a matter era followed by the accelerated expansion.

Necessary conditions were obtained in Amendola et al (2007), using dynamical system analysis.

3) Dolgov-Kawasaki instability (2003).

$$f(R) = R + \epsilon\varphi(R),$$

ε is a small parameter with dimension of mass^2, φ dimensionless.

$$3\Box f'(R) + f'(R)R - 2f(R) = \kappa T, \longrightarrow \Box R + \frac{\varphi'''}{\varphi''}\nabla^\alpha R \nabla_\alpha R + \frac{(\epsilon\varphi' - 1)}{3\epsilon\varphi''}R = \frac{\kappa T}{3\epsilon\varphi''} + \frac{2\varphi}{3\varphi''}.$$

Weak-field approx.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad R = -\kappa T + R_1, \qquad |R_1/\kappa T| \ll 1.$$

$$\ddot{R}_1 - \nabla^2 R_1 - \frac{2\kappa\varphi'''}{\varphi''}\dot{T}\dot{R}_1 + \frac{2\kappa\varphi'''}{\varphi''}\vec{\nabla}T \cdot \vec{\nabla}R_1 + \boxed{\frac{1}{3\varphi''}\left(\frac{1}{\epsilon} - \varphi'\right)}R_1 = \kappa\ddot{T} - \kappa\nabla^2 T - \frac{(\kappa T\varphi' + 2\varphi)}{3\varphi''},$$

(operators built with The R^3 metric, and $\varphi(R) = \varphi(-kT)$).

$$m_{eff}^2 \simeq (3\varepsilon\varphi'')^{-1} \Longrightarrow f''(R) > 0 .$$

There are several models that satisfy the appropriate criteria and are compatible with the weak field limit (more about this later):

$$f(R) = -m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1},$$

Hu y Sawicki (2007)

$$f(R) = R + \lambda R_0 \left(\left(1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right),$$

Starobinski (2007)

$$F(R) = \frac{1}{2} R + \frac{1}{2a} \log \left[\cosh(aR) - \tanh(b) \sinh(aR) \right].$$

Appleby y Battye (2007)

$$f(R) = -\alpha \left(\tanh\left(\frac{b\,(R - R_0)}{2} \right) + \tanh\left(\frac{bR_0}{2} \right) \right) = -\alpha \left(\frac{e^{b(R-R_0)} - 1}{e^{b(R-R_0)} + 1} + \frac{e^{bR_0} - 1}{e^{bR_0} + 1} \right)$$

Cognola et al (2008)

$$f(R) = -c\,r\,(1 - e^{-R/r}).$$

Linder (2009)

• *f(ℛ)* theories in the Palatini formalism

The fact that the result of the Palatini variation leads to a theory that is very different from that obtained by the metric variation in the case of f(R) theories was noticed first in

Buchdahl, H. A., 1970, Mon. Not. Roy. Astron. Soc. **150**, 1.

Given the action

$$S_{pal} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(\mathcal{R}) + S_M(g_{\mu\nu},\psi).$$

where $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ is built with the connection Γ, a priori independent of the metric, there are two cases:

a) S_M does not depend of Γ (the covariant derivative is built with the Christoffels) → We shall see that Γ is actually an auxiliary field.

b) S_M depends of Γ → metric-affine theories.

- **We shall see that in the case (a) of *f(ℛ)* theories in the Palatini formalism**

1) The metric is the sole field associated to gravity. ✔

2) Matter fields are universally coupled to the metric. ✔

3) The EOM are PDEs of 2nd order in the metric ✔, and do not display derivatives of the metric tensor. ×

***a) S_M* does not depend of Γ**

Taking the variation of the action wrt the metric and Γ :

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = 8\pi G T_{\mu\nu},$$
$$\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) = 0,$$

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$$

(with the covariant derivative constructed with Γ).

$f(\mathcal{R}) = \mathcal{R}$ → RG

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = 8\pi G T_{\mu\nu},$$ → $$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 8\pi G T.$$

No scalar mode.

$$\bar{\nabla}_\lambda \left(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}\right) = 0$$

$$h_{\mu\nu} \equiv f'(\mathcal{R}) g_{\mu\nu}.$$

$$\Gamma^\lambda{}_{\mu\nu} = h^{\lambda\sigma} \left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right)$$

The connection is symmetric , hence there is no torsion.

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{f'(\mathcal{R})} g^{\lambda\sigma} \left(\partial_\mu \left(f'(\mathcal{R}) g_{\nu\sigma}\right) + \partial_\nu \left(f'(\mathcal{R}) g_{\mu\sigma}\right) - \partial_\sigma \left(f'(\mathcal{R}) g_{\mu\nu}\right)\right)$$

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2}\frac{1}{[f'(\mathcal{R})]^2}[\nabla_\mu f'(\mathcal{R})][\nabla_\nu f'(\mathcal{R})] - \frac{1}{f'(\mathcal{R})}\left(\nabla_\mu \nabla_\nu - \frac{1}{2} g_{\mu\nu} \Box\right) f'(\mathcal{R}).$$

$$\mathcal{R} = R + \frac{3}{2[f'(\mathcal{R})]^2}[\nabla_\mu f'(\mathcal{R})][\nabla^\mu f'(\mathcal{R})] + \frac{3}{f'(\mathcal{R})} \Box f'(\mathcal{R}).$$

$$f'(\mathcal{R}) \mathcal{R}_{(\mu\nu)} - \tfrac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \kappa G T_{\mu\nu},$$

$$G_{\mu\nu} = \frac{8\pi G}{f'} T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(\mathcal{R} - \frac{f}{f'} \right) + \frac{1}{f'} \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) f' -$$
$$- \frac{3}{2} \frac{1}{f'^2} \left((\nabla_\mu f')(\nabla_\nu f') - \frac{1}{2} g_{\mu\nu} (\nabla f')^2 \right).$$

The result of the Palatini variation is a metric theory with 2nd order derivatives, with the energy-momentum tensor and 1st and 2nd derivatives of its components as a source, through the relation

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 8\pi G T. \quad \rightarrow \quad \mathcal{R} = \mathcal{R}(T)$$

b) S_M depends of Γ

$$S_{ma} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(\mathcal{R}) + S_M(g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \psi).$$

The ensuing theory is non-metric, and it may have non-zero non-dynamical torsion.

Sotiriou, T. P., and S. Liberati, 2007b, Annals Phys. **322**, 935.

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu},$$

$$\frac{1}{\sqrt{-g}}\left[-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right)\delta^{\nu}{}_{\lambda}\right] +$$

$$+2f'(\mathcal{R})g^{\mu\sigma}S_{\sigma\lambda}{}^{\nu} = \kappa(\Delta_{\lambda}{}^{\mu\nu} - \frac{2}{3}\Delta_{\sigma}{}^{\sigma[\nu}\delta^{\mu]}{}_{\lambda}),$$

$$S_{\mu\sigma}{}^{\sigma} = 0.$$

$$\Delta_{\lambda}{}^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta\Gamma^{\lambda}{}_{\mu\nu}},$$

Tensor de hiper-momento

Tensor de torsión

$$S_{\mu\nu}{}^{\lambda} \equiv \Gamma^{\lambda}{}_{[\mu\nu]}.$$

Conditions on $f(\mathcal{R})$

$$G_{\mu\nu} = \frac{\kappa}{f'}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(\mathcal{R} - \frac{f}{f'}\right) + \frac{1}{f'}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f' - \frac{3}{2}\frac{1}{f'^2}\left[(\nabla_\mu f')(\nabla_\nu f') - \frac{1}{2}g_{\mu\nu}(\nabla f')^2\right].$$

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa T.$$

$$\mathcal{R} = R + \frac{3}{2[f'(\mathcal{R})]^2}[\nabla_\mu f'(\mathcal{R})][\nabla^\mu f'(\mathcal{R})] + \frac{3}{f'(\mathcal{R})}\Box f'(\mathcal{R}).$$

→

1) **The coupling with matter is no longer controlled by *f '*.**

2) **$\mathcal{R}$ has no dynamics, hence there is no D-K instability.**

More general 4th order theories

$$\mathcal{L} = \chi^{-1}\sqrt{-g}f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$$

$$X \equiv R,\ Y \equiv R_{\mu\nu}R^{\mu\nu}$$

$$Z \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

$$P_{\mu\nu} = \frac{\chi}{2}T_{\mu\nu} - g_{\mu\nu}\Lambda$$

$$\begin{aligned} P^{\mu\nu} \quad \equiv \quad & -\frac{1}{2}fg^{\mu\nu} + f_X R^{\mu\nu} + 2f_Y R^{\rho(\mu}R^{\nu)}{}_{\rho} + 2f_Z R^{\epsilon\sigma\rho(\mu}R^{\nu)}{}_{\rho\sigma\epsilon} \\ & + f_{X;\rho\sigma}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}) + \Box(f_Y R^{\mu\nu}) + g^{\mu\nu}(f_Y R^{\rho\sigma})_{;\rho\sigma} \\ & -2(f_Y R^{\rho(\mu})_{;}{}^{\nu)}{}_{\rho} - 4(f_Z R^{\sigma(\mu\nu)\rho})_{;\rho\sigma}. \end{aligned}$$

(for the EOM in the Palatini formalism, see Li et al (2007)).

- **The resultant field equations are generically of 4th order.**
- **Some particular cases studied regarding accelerated expansion:**

$$L = R + \alpha\sqrt{R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}}$$

Uddin et al (2009)

$$L = R + \frac{\mu^{4n+2}}{(aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^n},$$

Carroll et al (2005)

$$L = R + f(\hat{G})$$

$$\hat{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

De Felice and Tsujikawa (2009)

(It yields 2nd order equations).

Yet another example (to show how short range forces look like)

$$I = -\int (-g)^{1/2} (\alpha R_{\mu\nu} R^{\mu\nu} - \beta R^2 + \gamma \kappa^{-2} R)$$

Stelle (1997)

$$H_{\mu\nu} = (\alpha - 2\beta) R_{;\mu;\nu} - \alpha R_{\mu\nu}{}^{;\eta}{}_{;\eta} - (\tfrac{1}{2}\alpha - 2\beta) g_{\mu\nu} R^{;\eta}{}_{;\eta} + 2\alpha R^{\eta\lambda} R_{\mu\eta\nu\lambda}$$
$$- 2\beta R R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} (\alpha R^{\eta\lambda} R_{\eta\lambda} - \beta R^2) + \gamma \kappa^{-2} R_{\mu\nu} - \tfrac{1}{2} \gamma \kappa^{-2} g_{\mu\nu} R = -\tfrac{1}{2} T_{\mu\nu}$$

$$H^{\mu\nu}{}_{;\nu} \equiv 0$$

$$ds^2 = A(r) dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) - B(r) dt^2$$

$$A(r) = 1 + W(r)$$

$$B(r) = 1 + V(r)$$

Linearizing the EOM in *V* and *W*, and solving for V for the source

$$T_{\mu\nu} = \delta^0_\mu \delta^0_\nu M \delta^3(x),$$

the weak-field potential is given by

$$V = \frac{-\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 M}{6\pi\gamma} \frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M}{24\pi\gamma} \frac{e^{-m_0 r}}{r}$$

$$m_2 = \gamma^{1/2}(\alpha\kappa^2)^{-1/2}, m_0 = \gamma^{1/2}[2(3\beta - \alpha)\kappa^2]^{-1/2}$$

(m_2 is the mass of a spin 2 field, and m_0, that of a scalar field).

- **There are extra degrees of freedom in the general 4th order theories (compared to the f(R) in the metric formalism).**

- **The modification of the Newtonian potential is very constrained by experiments.**

Other theories not covered here:

- **Hybrid *f (R, $\mathscr{R}$)* theories, S. Capozziello et al, Universe 1, 188 (2016).**
- ***f(T),* Cai et al, Rept. Prog. Phys., 79, 10, 106901 (2016).**
- **Einstein-æther theories, Jacobson & Speranza, PRD 92, 044030 (2015).**
- **TeVes, Saridakis, C&Q Grav., 26, 143001 (2009).**
- **Bi-metric theories, Schmidt-May & Von Strauss, 1512.00021 [hep-th].**
- **Brane worlds, Maartens & Koyama, Living Rev. Rel. 13, 5 (2010).**
- **...**