Cosmological solutions in f(T) and modified teleparallel theories of gravity

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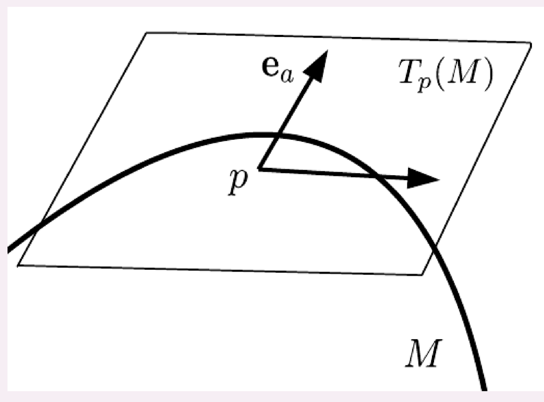
Abstract

f(T) gravity, a kind of modified teleparallel theory of gravity, has attracted a lot of attention in the recent years due to its hability to predict an early accelerated expansion of the Universe without resorting to an inflaton field, and since it allows to smooth black-hole singularities for modifications *à la* Born-Infeld. This theory is a generalization of the teleparallel equivalent of general relativity (TEGR), a dynamical theory for the tetrad or vielbein, based in the torsion of the Weitzenböck connection that describes a spacetime with zero curvature but non-vanishing torsion.

Teleparallel and $f(T)$ gravity

In teleparallel gravity (TG), the dynamical object is the tetrad field $\mathbf{e}_a = e_a^\mu \partial_\mu$, which is a basis of vectors in the tangent space of a manifold. The co-tetrad E^a is given by $\mathbf{E}^a = E^a_\mu dx^\mu$. From the co-tetrad components it can be obtained the metric $\mathbf{g}$ through the relation



$$g_{\mu\nu} = \eta_{ab} E^a_\mu E^b_\nu,$$

with $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$ the Minkowski metric, and where it is also that $\sqrt{-g} = \det[E^a_\mu] = E$. In TG we work with the torsion tensor $T^\mu{}_{\nu\rho} = e^\mu{}_a(\partial_\nu E^a{}_\rho - \partial_\rho E^a{}_\nu)$, which can be regarded as the torsion of the Weitzenböck connection $\overset{\mathrm{W}}{\Gamma}{}^\mu{}_{\rho\nu} = e^\mu{}_a \partial_\nu E^a{}_\rho$. The action of TG is given by

$$S = \frac{1}{2\kappa}\int d^4x\, e S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} = \frac{1}{2\kappa}\int d^4x\, eT,$$

where T is the torsion scalar $T = S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}$, and $2S_\rho{}^{\mu\nu} = K^{\mu\nu}{}_\rho + T_\lambda{}^{\lambda\mu}\delta^\nu_\rho - T_\lambda{}^{\lambda\nu}\delta^\mu_\rho$, with $K^\lambda{}_{\mu\nu} = \overset{\mathrm{W}}{\Gamma}{}^\lambda{}_{\mu\nu} - \overset{\mathrm{LC}}{\Gamma}{}^\lambda{}_{\mu\nu}$ as the contorsion tensor. The equivalence between TG and GR comes from the identity

$$-ER = ET - 2\partial_\rho(ET_\mu{}^{\mu\rho}), \qquad (1)$$

which states that their Lagrangians differ only in a 4-divergence that is only global Lorentz invariant. TG globally determines the field of tetrads providing the spacetime a metric with an absolute parallelization. The simplest generalization of the TG action is the so-called $f(T)$ gravity, whose action is given by [1,2]

$$S = \frac{1}{2\kappa}\int d^4x E f(T). \qquad (2)$$

Since the most general f is not linear in T, the 4-divergence in (1) remains encapsulated inside the functional form, leaving the theory only globally Lorentz invariant except for a remnant symmetry group of local Lorentz transformations [1,3,4]. Recently it was shown that the local Lorentz invariance is lost in only one generator of Lorentz transformations, that could be a combination of boosts and rotations that are fixed by the theory [6,8].

Vacuum solutions in $f(T)$

The equations of motion of $f(T)$ gravity are obtained by varying the action (2) with respect to the tetrad E^a_μ, obtaining

$$\left(e_a^\lambda T^\rho{}_{\mu\lambda} S_\rho{}^{\mu\nu} + E^{-1}\partial_\mu(E S_a{}^{\mu\nu})\right) f' + S_a{}^{\mu\nu}\partial_\mu T f'' - \frac{e_a^\nu}{4} f = -\frac{\kappa}{2} e_a^\lambda \mathcal{T}_\lambda^\nu,$$

where $\mathcal{T}_\mu^\nu$ is the energy-momentum tensor. An easy way to obtain solutions for these e.o.m. is to obtain a torsion scalar that is constant ($T = T_c$). Then $\partial_\mu T = 0$, and the dynamical equations can be arranged as [3]

$$G_\mu^\nu + \frac{\delta_\mu^\nu}{2}\left(\frac{f(T_c)}{f'(T_c)} - T_c\right) = \frac{\kappa \mathcal{T}_\mu^\nu}{f'(T_c)}, \qquad (3)$$

where $G_\mu^\nu = -2e^a_\mu(e_a^\lambda T^\rho{}_{\sigma\lambda} S_\rho{}^{\sigma\nu} + E^{-1}\partial_\sigma(E S_a{}^{\sigma\nu})) + \frac{1}{2}\delta^\nu_\mu T$ is the Einstein tensor. Notably the e.o.m. (3) are the Einstein equations with with a scaled Newton constant $\tilde{G} = G/f'(T_c)$ and redefined cosmological constant $\Lambda = \frac{1}{2}\left(T_c - \frac{f(T_c)}{f'(T_c)}\right)$.

McVittie geometry

The McVittie geometry describes a black hole solution embedded in an expanding FLRW universe. The McVittie metric can be written as

$$ds^2 = \left(1 - \frac{2m}{R} - H(t)^2R^2\right)dt^2 + \frac{2H(t)R}{\sqrt{1-2m/R}}dRdt - \frac{dR^2}{1-2m/R} - R^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and $\mathbf{R} = (1+\mu)^2 a(t)\mathbf{x}$ is a radial coordinate which represents a "spherical area" coordinate.

We use a null tetrad approach to find a suitable tetrad that describes the McVittie metric and that is solution to $f(T)$ dynamical equations [5,7]. For this, we define a null tetrad $\{\mathbf{n}^a\} = \{\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}}\}$ in terms of an orthonormal tetrad $\{\mathbf{e}^a\} = \{\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ as $\{\mathbf{n}^a\} = \frac{1}{\sqrt{2}}\{\mathbf{e}^0 - \mathbf{e}^1, \mathbf{e}^0 + \mathbf{e}^1, \mathbf{e}^2 + i\mathbf{e}^3, \mathbf{e}^2 - i\mathbf{e}^3\}$. It is straightforward to prove that the metric tensor can be obtained through this null tetrad from the expressions

$$g_{\mu\nu} = \eta_{ab} n^a_\mu n^b_\nu, \quad \eta_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

A good strategy to incorporate a local Lorentz boost along the direction of $\mathbf{e}^1$ is to perform the transformation

$$\{\mathbf{l}, \mathbf{n}\} \longrightarrow \{\exp[-\lambda(x)]\mathbf{l}, \exp[\lambda(x)]\mathbf{n}\},$$

which leaves the metric unchanged but introduces an additional degree of freedom λ which can be adjusted to find solutions with $T = 0$. In particular, we find that the following null tetrad

$$n^a_\mu = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\lambda}(\gamma + RH) & -\frac{e^{-\lambda}}{\gamma} & 0 & 0 \\ e^{\lambda}(\gamma - RH) & \frac{e^{\lambda}}{\gamma} & 0 & 0 \\ 0 & 0 & R & iR\sin(\theta) \\ 0 & 0 & R & -iR\sin(\theta) \end{pmatrix}$$

where $\gamma = \sqrt{1 - \frac{2m}{R}}$, satisfies $T = 0$ for the choice $\lambda = \frac{t}{2R} - \frac{3R}{2}\int H^2(t)dt$ [7].

Tetrads for FLRW metric

$f(T)$ equations admit the FLRW metric as a solution with the diagonal tetrad [2,3]

$$e^0 = dt, \; e^1 = a(t)dx, \; e^2 = a(t)dy, \; e^3 = a(t)dz,$$

for a perfect fluid source with pressure p and energy density ρ. This tetrad has $T_H = -6H^2$, and the Friedman equations read

$$12H^2 f'(T_H) + f(T_H) = 2\kappa\rho,$$
$$48H^2 f''(T_H)\dot{H} - f'(T_H)(12H^2 + 4\dot{H}) - f(T_H) = 2\kappa p.$$

These equations show a term that behaves like dark energy for a particular choice of f. In particular, for $f(-6H^2) = -6H^2 - \alpha/(-6H^2)^n$ the Friedmann equation becomes [3]

$$H^2 - \frac{(2n+1)\alpha}{6^{n+1}H^{2n}} = \frac{\kappa}{3}\rho.$$

The second term in the l.h.s. represents a dark energy-like contribution to the Friedmann equations, due to the geometrical deformation induced by the $f(T)$ modification of TEGR. From the tetrad obtained for the McVittie metric it is possible to find another cosmological tetrad with $T_0 = 0$ that solves the equations of motion of $f(T)$. This is easy to see since in the limit $m = 0$, the McVittie geometry reduces to the FLRW metric. The null tetrad obtained is [7]

$$n^a_\mu = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\lambda}(1+RH) & -e^{-\lambda} & 0 & 0 \\ e^{\lambda}(1-RH) & e^{\lambda} & 0 & 0 \\ 0 & 0 & R & iR\sin(\theta) \\ 0 & 0 & R & -iR\sin(\theta) \end{pmatrix}$$

where λ has the same value than for the McVittie solution. This tetrad has also consistent equations of motion, given by

$$f(T_0) + 6H^2 f'(T_0) = 2\kappa\rho,$$
$$-4f'(T_0)\dot{H} = 2\kappa(p+\rho).$$

We remark the fact that the value of λ does not depends on the black hole mass m, even in the former McVittie case. The fact that there are two tetrads for the same metric with different torsion scalar could be a manifestation of the additional degree of freedom of the theory [8].

Conclusions

We obtain the McVittie geometry as a solution in $f(T)$ gravity. For this we introduce a null tetrad approach that facilitates the search for solutions of the e.o.m.. By taking $m = 0$ we also obtain a new tetrad that represents a FLRW universe which has $T = 0$. This result contrast with other cosmological solution having $T = -6H^2$. More work needs to be done regarding the stability of both solutions and its relation with the Lorentz breaking mechanism.

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